ENERGY METHOD AND LOCALIZATION OF SOLUTIONS OF EQUATIONS OF CONTINUUM MECHANICS

S. N. Antontsev and J. I. Diaz

UDC 532.5+517.946

Many important mathematical models in continuum mechanics lead to the study of nonlinear systems of differential equations with partial derivatives of the composite type. In such systems, different components of the sought vector-solution (such as velocity, density, pressure, saturation, temperature, etc.) satisfy equations of different types (parabolic, hyperbolic, elliptic). The systems of equations are degenerate in the sense of their type or order for certain values of the sought solution or its derivatives. Here, the solutions themselves have a finite time of localization (vanishing), a finite rate of propagation of perturbations from the initial data, spatial localization with inertia (metastable), etc. The fact that the solutions of degenerate parabolic equations (equations of the type describing nonlinear heat conduction) are associated with a finite perturbation velocity was evidently first noted and studied in [1-3], while the same property was discovered and investigated for elliptic equations in [4] in connection with study of the problem of the discharge of a plane sonic jet. A large number of studies was subsequently devoted to examining these questions for a single parabolic equation. A fairly complete survey of these studies can be found in [5, 6]. Questions related to the localization of solutions which increase without limit during a finite time are now being actively investigated for quasilinear parabolic equations [7]. The results for one parabolic equation have generally been obtained on the basis of theorems which compare the test solution with an auxiliary solution, such as a similarity solution. Methods of this type generally cannot be used for systems of equations of the composite type. The authors of [8-10] proposed and substantiated an energy method of studying the character of perturbations described by the solutions of general equations of the elliptic, parabolic, and composite types. The method is based on deriving and studying ordinary differential inequalities for energy functions. The method was generalized and developed in [11-18] to embrace higher-order equations. It has proven effective for investigating weak generalized solutions of systems of the composite type encountered in continuum mechanics.

In [11, 19-23], the energy method was used to establish the finite time of localization and finite rate of propagation of perturbations from initial data in several mathematical models of continuum mechanics (filtration flows of a two-phase liquid, combined flows of surface water and groundwater, flows of water in open channels, flows of incompressible nonuniform viscoplastic media, unidimensional flows of viscous gas, etc.). It was shown in [21] that an axisymmetric jet moving along the symmetry axis at sonic velocity levels out (similar to the plane case [4]) over a finite distance.

Here, we establish the finite localization time and metastable localization (localization with inertia) of solutions for some of the above-mentioned models in the presence of "sources" – assigned right sides. It should be noted that we will not address questions relating to the existence of the corresponding solutions. We will study only their qualitative properties.

1. Incompressible Nonuniform Non-Newtonian Fluids. The system of equations representing the laws of conservation is of the composite type and can be written in the form [24-26]

$$\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho = 0, \quad \nabla\rho = \left(\frac{\partial\rho}{\partial x_1}, \ldots, \frac{\partial\rho}{\partial x_n}\right); \tag{1.1}$$

$$\operatorname{div} \mathbf{v} = 0, \ \mathbf{v} = (v_1, \ldots, v_n); \tag{1.2}$$

$$\rho\frac{d\mathbf{v}}{dt} \equiv \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right) = \operatorname{div} P + \rho\mathbf{f}; \tag{1.3}$$

Novosibirsk, Madrid. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 18-25, March-April, 1989. Original article submitted August 11, 1988.

$$P = -pE + F(D), \quad D = \{D_{ij}\} = \left\{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right\}, \tag{1.4}$$

$$x \in \Omega \subset R^n, \ t \in (0, T), \ Q = \Omega \times (0, T).$$

Here, $\mathbf{v}(t, x)$, $\rho(t, x)$, and $p(t, x)$ are the sought velocity, density, and pressure in the fluid; P and D are the stress tensor and strain-rate tensor; E is the unit tensor; $\mathbf{f}(t, x)$ is the prescribed body force — a "source."

We will assume that the assigned symmetric tensor F, determining P, satisfies the condition

$$\delta |D|^q \leqslant F : D = F_{ij}D_{ij}, \ 1 < q, \ \delta = \text{const} > 0. \tag{1.5}$$

For the classical incompressible viscous fluid, $P = -pE + 2\mu D$, and, in (1.5), $\delta = 2$, $q = 2$. For viscoplastic fluids [25, 26], $P = -pE + 2(\mu + \tau|D|^{\sigma-1})D$, $0 \leq \sigma < 1$ and, using the Young inequality, we have (1.5) with

$$\delta = \mu^{1/\Theta}\tau^{(\Theta-1)/\Theta}\Theta^{1/\Theta}(\Theta/\Theta - 1)^{(\Theta-1)/\Theta},$$
$$q = \{2/\Theta + [(\sigma + 1)(\Theta - 1)]/\Theta\} \in (\sigma + 1, 2). \tag{1.6}$$

Let us examine the following initial-boundary-value problem for $w = (\mathbf{v}, \rho, p)$

$$\mathbf{v}(t, x) = 0, \ (x, t) \in \Gamma_T = \partial\Omega \times (0, T); \tag{1.7}$$

$$\mathbf{v}(0, x) = v_0(x), \ \rho(0, x) = \rho_0(x), \ x \in \Omega. \tag{1.8}$$

It should be noted that for problem (1.1)-(1.4), (1.7), (1.8) in Eq. (1.5) with $\sigma = 0$ the author of [27] proved the theorem of the existence of the weak generalized solution $w = (\mathbf{v}, \rho, p) \in V_q$, where $V_q = \{w: \ \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W_0^{1,q}(\Omega)), 1/M \leq \rho \leq M, \rho_t \in L^2(0, T; W^{-1,2}(\Omega))\}$ $(q = 2)$, with allowance for the notation in [28]. For a uniform incompressible fluid [$\rho(t, x) \equiv$ const], theorems of the existence of the solution of system (1.1)-(1.4) for certain relations (1.5) were proven in [28, 29]. We will study qualitative properties of the solutions $w \in V_q$ of system (1.1)-(1.4), assuming that Ω is a finite region with a smooth boundary. Let condition (1.5) be satisfied. In this condition

$$q \in (2n/(2 + n), 2), \ n \geqslant 2, \tag{1.9}$$

and, additionally,

$$\|\mathbf{v}_0(x)\|_{2,\Omega} \leqslant C_v = \text{const}, \ 1/M \leqslant \rho_0 \leqslant M; \tag{1.10}$$

$$\|\mathbf{f}(t, \cdot)\|_{2,\Omega}^{q/(q-1)} \leqslant C_f (1 - t/T_f)_+^{q/(2-q)},$$
$$C_f = \text{const}, \ u_+ = \max(0, u), \ T_f \in (0, T). \tag{1.11}$$

Theorem 1.1 (finite localization time). Let $w = (\mathbf{v}, \rho, p) \in V_q$ be the generalized solution of problem (1.1)-(1.4), (1.7), (1.8) and let conditions (1.5), (1.9)-(1.11) be satisfied. Then for any $T_f \in (0, T)$ there exist constants C_v, C_f (generally small relative to δ) and C such that

$$\|\mathbf{v}(t, \cdot)\|_{2,\Omega}^2 \leqslant C(1 - t/T_f)_+^{q/(2-q)} \tag{1.12}$$

and, in particular,

$$\mathbf{v}(t, x) \equiv 0, \ x \in \Omega, \ T_f \leqslant t. \tag{1.13}$$

Proof. Following the method of energy estimates [9, 11-13], we first prove the following equality for the solution w being examined:

$$\frac{1}{2}\frac{d\Pi}{dt} = (\operatorname{div} P, \mathbf{v})_\Omega + (\rho f, \mathbf{v})_\Omega = -(F: D, 1)_\Omega + (\rho \mathbf{f}, \mathbf{v})_\Omega \equiv I,$$
$$\Pi(t) = (\rho(t, \cdot)\mathbf{v}(t, \cdot), \mathbf{v}(t, \cdot))_\Omega, \quad (u, v)_\Omega = \int_\Omega uv\,dx. \tag{1.14}$$

The latter is formally obtained by multiplying Eq. (1.3) by $\mathbf{v}(t, x)$ and then integrating by parts with allowance for Eqs. (1.1), (1.2) and boundary condition (1.7).

We then use the Cornu inequality [26]

$$K\|\mathbf{v}\|_{m,\Omega} \leqslant \|D(\mathbf{v})\|_{q,\Omega},\ m \leqslant qn/(n-q) \tag{1.15}$$

with m = 2 and with allowance for (1.5), (1.11) to evaluate the right side of (1.14) as follows:

$$\begin{gathered} I \leqslant -\delta\|D\|_{q,\Omega}^q + MK^{-1}\|\mathbf{f}\|_{2,\Omega}\|D\|_{q,\Omega} \leqslant \\ \leqslant \frac{1}{2}\left(-a\Pi^{q/2} + b(1-t/T_f)_+^{q/(2-q)}\right), \\ a = \delta(K/\sqrt{M})^q,\ b = (M/q)^{q/(q-1)}(q-1)/q(\delta q/2)^{-1(q-1)}C_f. \end{gathered} \tag{1.16}$$

Combining (1.14) and (1.16), we arrive at an ordinary differential inequality for the energy function $\Pi(t)$:

$$d\Pi/dt + a\Pi^{q/2} \leqslant b(1-t/T_f)_+^{q/(2-q)},\quad \Pi(0) \leqslant MC_v. \tag{1.17}$$

All of the nonnegative solutions of inequality (1.17) are dominated by the function $\bar{\Pi}(t) = MC_v(1 - t/T_f)^{2/(2-q)}$ if the constants C_n, C_f, T_f, M, K, q, and δ satisfy the relation $-2MC_v/(2-q)T_f + a(MC_v)^{q/2} \geq b$. Inspection of the latter completes the proof of the theorem.

Note 1.1. Theorem 1.1 has the following physical interpretation. The flow of a non-Newtonian fluid [with conditions (1.5), (1.9)], initiated by the initial data and body forces (the "source"), begins from a state of rest $v \equiv 0$ at the moment of time T_f — the connection of the "sources."

Note 1.2. Theorem 1.1 can also be formulated in the following manner: for any constant $C_v \in (0, \infty)$ in (1.10) and sufficiently small C_f in (1.11), there exists $T_f \in (0, \infty)$ such that (1.12) and (1.13) are valid. A similar theorem for $f \equiv 0$ was proven in [11].

Note 1.3. The constant K in inequality (1.15) is independent of Ω if m = 2, q = 2n/(n + 2), q > 1. Thus, in the present case, the above-formulated theorems are also valid for the Cauchy problem $v(0, x) = v_0(x)$, $\rho(0, x) = \rho_0(x)$, $x \in R^n$ for system (1.1)-(1.4).

Now let us study local properties of the solutions of system (1.1)-(1.4) outside of the connection with the boundary conditions. We will restrict ourselves to examining solutions of the particular form

$$\begin{gathered} \mathbf{v}(t,x) = (0,\ 0,\ w(t,x_1,x_2)),\ \rho(t,x) = 1, \\ \mathbf{f}(t,x) = (0,\ 0,\ f(t,x_1,x_2)),\ \partial p/\partial x_3 = a(t), \end{gathered} \tag{1.18}$$

assuming that the pressure gradient $a(t)$ is assigned. This solution can correspond to flow in a pipe. Then we write system (1.1)-(1.4) in the form

$$\partial w/\partial t = \operatorname{div}\mathbf{F}(\nabla w) - \partial p/\partial x_3 + f. \tag{1.19}$$

We assume that the vector $\mathbf{F}(\nabla w)$ satisfies the condition

$$\delta|\nabla w|^q \leqslant \mathbf{F}(\nabla w)\nabla w \leqslant \delta^{-1}|\nabla w|^q,\ 2 < q. \tag{1.20}$$

We introduce the notation $B_\rho(x_0) = \{x\colon\ x \in \Omega,\ |x - x_0| < \rho\}$ and, in the region $B_{\rho_1} \times (0, T)$, we examine the solution of Eq. (1.19) with the initial condition

$$w(0,x) = w_0(x),\quad x \in B_{\rho_1}. \tag{1.21}$$

It is further assumed that

$$\left(\|w_0\|_{2,B_\rho}^2 + \int_0^T \|f(\tau,\cdot)\|_{2,B_\rho}^2 d\tau\right) \leqslant C(\rho-\rho_0)_+^{1/(1-\alpha)}, \tag{1.22}$$

$$\begin{gathered} \rho \in (0,\rho_1),\ 0 < \rho_0 < \rho_1,\ \alpha = (3q-2)/4(q-1); \\ \left(|a(t)| + \|w(t,\cdot)\|_{2,B_{\rho_1}}^2\right) \leqslant M. \end{gathered} \tag{1.23}$$

THEOREM 1.2 (metastable localization). Let $w(t, x) \in V_q$ be the generalized solution of Eq. (1.19) in $B_{\rho_1} \times (0, T)$ with initial condition (1.21), and let conditions (1.20)-(1.23) be satisfied. Then there exists $t_0 = t_0(M, q, \rho_1, \delta) \in (0, T)$ such that

$$u(t, x) = \left(w(t, x) + \int_0^t a(\tau)\, d\tau \right) = 0, \quad x \in B_{\rho_0}, \quad 0 \leqslant t \leqslant t_0. \tag{1.24}$$

Proof. We introduce the energy functions

$$\Pi(t, \rho) = (u(t, \cdot), u(t, \cdot))_{B_\rho}, \quad b(t, \rho) = \sup_{0 \leqslant \tau \leqslant t} \Pi(\tau, \rho),$$
$$E(t, \rho) = \int_0^t (\mathbf{F}(\nabla u), \nabla u)_{B_\rho} d\tau, \tag{1.25}$$

which, as can readily be shown, have the properties

$$\frac{\partial E}{\partial \rho} = \int_0^t (\mathbf{F}(\nabla u), \nabla u)_{\partial B_\rho} d\tau \geqslant \delta \int_0^t \| \nabla u \|^q_{q, \partial B_\rho} d\tau,$$
$$\frac{\partial E}{\partial t} = (F, \nabla u)_{B_\rho} \geqslant \delta \| \nabla u \|^q_{q, B_\rho}. \tag{1.26}$$

In accordance with Eq. (1.19), for the function $u(t, x) = w(t, x) + \int_0^t a(\tau)\, d\tau$, we obtain the energy equality

$$\frac{1}{2}(\Pi(t, \rho) - \Pi(0, \rho)) + E(t, \rho) = I_1 + I_2, \tag{1.27}$$

where $I_1 = \int_0^t (\mathbf{F} \cdot \mathbf{n}, u)_{\partial B_\rho} d\tau; I_2 = \int_0^t (f, u)_{B_\rho} d\tau$: $\mathbf{n}$ is the vector of the normal to ∂B_ρ. In accordance with [11-13], the terms in the right side of (1.27) can be evaluated as follows:

$$|I_1| \leqslant \frac{\varepsilon}{2} Tb(t, \rho) + \frac{1}{2\varepsilon} \int_0^t \| f(\tau, \cdot) \|^2_{2, B_\rho} d\tau, \quad \varepsilon > 0,$$
$$|I_2| \leqslant \frac{1}{\delta} \int_0^t \| \nabla u \|^q_{q, \partial B_\rho} \| u \|_{q, \partial B_\rho} d\tau \leqslant \varepsilon (E + b) + C t^\gamma \left(\frac{\partial E}{\partial \rho} \right)^{1/\alpha} \tag{1.28}$$
$$(C = C(T, M, q, \delta, \varepsilon),\ \gamma = 8/(3q-1),\ \alpha = 3q - 2/4(q-1)).$$

Combining (1.27) and (1.28), allowing for (1.22), and appropriately choosing $\varepsilon > 0$, we arrive at the final inequality

$$E \leqslant E + b \leqslant a t^\gamma (\partial E / \partial \rho)^{1/\alpha} + b (\rho - \rho_0)_+^{1/(1-\alpha)}, \tag{1.29}$$

in which a and b depend only on M, q, δ, and T. In accordance with the results in [30, 31], for an inequality of the type (1.29) there exists $t_0 > 0$ such that $E(t, \rho_0) = 0$, $t \leq t \leq t_0$. This completes the proof of the theorem.

Note 1.4. Thus, if a non-Newtonian fluid [with the law (1.20)] is at rest [$w(0, x) = u(0, x) = 0$] in the region B_{ρ_0} at $t = 0$, then its motion is determined by the following relation, regardless of the boundary conditions and the "sources" outside B_{ρ_0}

$$w(t, x) = -\int_0^t a(\tau)\, d\tau, \quad 0 \leqslant t \leqslant t_0, \ x \in B_{\rho_0}.$$

In particular, the state of rest is maintained [$w(t, x) = 0$] at $t \in [0, t_0]$ if there is no pressure gradient ($a = 0$).

Note 1.5. As in the previous case, the energy method can be used to study problem (1.1)-(1.4) with allowance for the change in the temperature of the medium $\Theta(t, x)$ by adding the following equation to the system

$$\rho \frac{d\Theta}{dt} \equiv \rho \left(\frac{\partial \Theta}{\partial t} + (\mathbf{v} \nabla) \Theta \right) = \operatorname{div} \mathbf{A}(t, x, \Theta, \nabla \Theta) + L(t, x, \Theta, v).$$

Here, it is possible to consider the more general relation F(D, Θ), as well as nonlinear laws governing heat conduction and volumetric absorption in the form [11-13]

$$\delta_1|\Theta|^{\alpha}|\nabla\Theta|^{\varkappa} \leqslant A\nabla\Theta \leqslant (1/\delta_1)|\Theta|^{\alpha}|\nabla\Theta|^{\varkappa}, \quad -1 < \Theta, \ 1 < \varkappa,$$
$$L = -\gamma_0|\Theta|^{\sigma-1}\Theta + L_0(t, x, v), \quad 0 \leqslant \gamma, \ 0 < \sigma \leqslant 1.$$

2. Simultaneous Flows of Surface Water and Groundwater. The studies [11, 23] examined mathematical models of the simultaneous no-head flows of surface and subsurface water based on equations of plane filtration and the hydraulics of open channels.

In the simplest case (channel of rectangular cross section and constant width, water-confining stratum and channel bottom horizontal, etc.), the corresponding system of equations and the internal contact conditions have the form [23]

$$\frac{\partial H}{\partial t} = \mathrm{div}\,(H\nabla H) + f_\Omega(t, x), \quad x \in \Omega \setminus \Gamma; \tag{2.1}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial s}\left(\psi(s, u)\,|u_s|^{-1/2}\frac{\partial u}{\partial s}\right) - \left[H\frac{\partial H}{\partial n}\right]_\Gamma + f_\Gamma(t, x), \quad x \in \Gamma; \tag{2.2}$$

$$H\frac{\partial H}{\partial n}\Big|_{\Gamma_\pm} = \sigma_\pm(u - H_\pm), \quad x \in \Gamma, \quad 0 < \sigma_\pm = \mathrm{const}. \tag{2.3}$$

Here, H(t, x) is the level of the groundwater in the region $\Omega \subset R^2$; u(t, s) is the level of water in the channel, corresponding to the curve of Γ in Ω; s is the length of an arc along Γ; n is the vector of the normal to Γ; $H_\pm$ are the values of H in the approach of Γ from different directions (accordingly, $[H]_\Gamma = H_+ - H_-$); $f_\Omega(t, x)$, $F_\Gamma(t, x)$ are assigned external inflows of water – "sources."

With $f_\Gamma = f_\Omega = 0$, the authors of [11, 23] used the energy method to prove the finite velocity of the perturbations for H(t, x), u(t, x) from zero initial data. Below, we prove the existence of metastable localization for solutions (2.1)-(2.3).

We will study the local properties of the solution w = (H(t, x), u(t, x)) of system (2.1)-(2.3) in the circle $B_{\rho_1}(x_0) = \{x: x \in \Omega, |x - x_0| < \rho\}$, $x_0 \in \Gamma$ without restrictions on generality, assuming that $x_0 = 0$,

$$\Gamma_\rho = \{x\colon x \in \Omega, x_2 = 0, |x_1| < \rho\}, \quad s = x_1, \quad B_\rho^\pm = \{x\colon x \in B_\rho, 0 \lessgtr x_2\}.$$

The existence of the generalized solution w = (H, u) ∈ V was proven for system (2.1)-(2.3) in [23] for fundamental initial-boundary-value problems. Here,

$$V = \{(H, u)\colon 0 \leqslant (H, u) \leqslant M, \ \sqrt{H}\,\nabla H \in L^2(0, T; L^2(B_{\rho_1})),$$
$$\psi^{2/3}|u_s| \in L^{3/2}\left(0, T; L^{3/2}(\Gamma_{\rho_1})\right)\}, \quad (|\ln(\psi u^{-5/3})| \leqslant M).$$

THEOREM 2.1 (metastable localization). Let w = (H, u) ∈ V be the generalized solution of system (2.1)-(2.3) in $B_{\rho_1} \times (0, T)$ and

$$\left(\|H(0, \cdot)\|^2_{2,B_\rho^\pm} + \|u(0, \cdot)\|^2_{2,\Gamma_\rho} + \int_0^T \left(\|f_\Omega\|^2_{2,B_\rho^\pm} + \|f_\Gamma\|^2_{2,\Gamma_\rho}\right) d\tau\right) \leqslant$$
$$\leqslant C(\rho - \rho_0)_+^{1/(1-\alpha)}, \quad \rho \in (0, \rho_1), \quad 0 < \rho_0 < \rho_1, \quad \alpha = 5/6. \tag{2.4}$$

Then there exists $t_0 = t_0(M, C, \rho_1, T)$ such that $w = (H, u) = 0$, $x \in B_{\rho_0}$, $0 \le t \le t_0$.

Proof. We introduce the notation

$$\Pi(t, \rho) = \left(\|H(t, \cdot)\|^2_{2,B_\rho} + \|u(t, \cdot)\|^2_{2,\Gamma_\rho}\right), \quad b(t, \rho) = \sup_{0 \leqslant \tau \leqslant t} \Pi(\tau, \rho),$$

$$E(t, \rho) = \int_0^t \left((H\nabla H, \nabla H)_{B_\rho} + (\psi, |u_s|^{3/2})_{\Gamma_\rho}\right) d\tau,$$

$$D^2 = \int_0^t \sum_\pm (\sigma_\pm, (u - H_\pm)^2)_{\Gamma_\rho}\, d\tau, \quad F = \int_0^t \left((f_\Omega, H)_{B_\rho} + (f_\Gamma, u)_{\Gamma_\rho}\right) d\tau.$$

Then the energy equation corresponding to system (2.1)-(2.3) has the form

$$\frac{1}{2}(\Pi(t,\rho)-\Pi(0,\rho))+E(t,\rho)+D^2= \\ =F+\int_0^t\left(\left(H\frac{\partial H}{\partial n},H\right)_{\partial B_\rho}+\psi|u_s|^{-1/2}u_s u\Big|_{x_1=-\rho}^{x_1=\rho}\right)d\tau. \tag{2.5}$$

Performing calculations analogous to (1.28), we arrive at an inequality of the type (1.29). Analysis of the latter completes the proof of the theorem.

Note 2.1. Theorem 2.1 has the following physical interpretation. The region B_{ρ_0}, not occupied by water at the initial moment of time $t = 0$, remains so at $t \le t_0$ as well, regardless of the boundary conditions and the sources outside B_{ρ_0}.

3. Two-Phase Filtration of Immiscible Incompressible Fluids. The nonsteady filtration of two immiscible incompressible fluids in a nonuniform anisotropic medium is described by a system of equations of the composite type [22]:

$$m(x)\frac{\partial s}{\partial t}=\operatorname{div}(K_0(x)a(s)\nabla s-b(s)\mathbf{v}+\mathbf{F}(x,s)); \tag{3.1}$$

$$\operatorname{div}\mathbf{v}=0,\ \mathbf{v}=-(K(x,s)\nabla p+\mathbf{f}(x,s)),\ x\in\Omega\subset R^n. \tag{3.2}$$

Here, the sought functions are the saturation $s(t, x)$ $(0 \le s \le 1)$, the "corrected" pressure $p(t, x)$, and the velocity of the mixture $\mathbf{v}$. The coefficients of system (3.1)-(3.2) are determined by the formulas

$$a(s)=k_{01}k_{02}\left|\frac{\partial p_k}{\partial s}\right|\Big/(k_{01}+k_{02}),\quad b=k_{01}/k, \\ F=-\frac{k_{01}k_{02}}{k}K_0(\nabla_x p_k+(\rho_2-\rho_1)g),\quad k=k_{01}+k_{02}, \tag{3.3}$$

where p_k is the capillary pressure; $\mu_i k_{0i}$ are the relative phase permeabilities; μ_i and ρ_i are the viscosities and densities of the fluids; g is acceleration due to gravity; K_0 is the symmetric filtration tensor for a uniform fluid; m is porosity. Depending on the form of the functional parameter k_{01} and p_k, the coefficient $a(s)$ in (3.1) may either vanish or become infinite at values of $s = 0.1$ — thereby establishing the different character of propagation of saturation perturbations $s(t, x)$. For system (3.1)-(3.2), the authors of [22] proved the theorem of the existence of the generalized solution $w = (s, p) \in V$, where $V = \{(s, p): 0 \le s \le 1, \sqrt{a}\times\nabla s\in L^2(0, T; L^2(\Omega)), \nabla p\in L^\infty(0, T; L^q(\Omega))\}$, $n < q \le \infty$, while the authors of [10, 11] established a finite localization time $s(t, x)$ $[a(0) = \infty, s = 0]$ in the case of a boundary-value problem and a finite rate of propagation of perturbations from the initial data $[s(0, x) = 0$ or $s(0, x) = 1$, $a(0) = a(1) = 0]$. Below, we show that with the additional condition for the initial data $s(0, x) = s_0(x)$, this solution also has the property of metastable localization (localization with inertia) at $a(0) = 0$.

Let us examine system (3.1)-(3.2) in the region $B\rho_1 \times (0, T)$, assuming satisfaction of the conditions

$$M^{-1}\leqslant\left(m;\frac{(K_0\xi,\xi)}{(\xi,\xi)};\ k;\ \left|\frac{\partial p_k}{\partial s}\right|\right)\leqslant M,\quad 0\leqslant s\leqslant 1; \tag{3.4}$$

$$(|\ln(as^{-\alpha})|;\ |\mathbf{F}_s'|s^{1-\alpha};\ |\operatorname{div}_x\mathbf{F}|s^\alpha;\ |b_s'|s^{-(\alpha+\nu)/2})\leqslant M; \tag{3.5}$$

$$\|s(0,x)\|^2_{2,B_{\rho_0}}\leqslant C(\rho-\rho_0)_+^{1/(1-\alpha)},\ 2+\alpha n/q\leqslant\nu+2\leqslant\alpha. \tag{3.6}$$

THEOREM 3.1 (metastable localization). Let $w = (s, p) \in V$ be the generalized solution of metastable system (3.1)-(3.2) and let conditions (3.4)-(3.5) be satisfied. Then there exists $t_0 = t_0(M, C, \alpha, q, \rho_1) > 0$ such that

$$s(t,x)=0,\ x\in B_{\rho_0},\ 0\leqslant t\leqslant t_0.$$

Proof. We introduce the energy functions

$$\Pi(t,\rho)=(ms(t,\cdot),s(t,\cdot))_{B_\rho},\quad E(t,\rho)=\int_0^t(K_0 a\nabla s,\nabla s)_{B_\rho}d\tau$$

and we make use of Eq. (3.1) in the form

$$m\frac{\partial s}{\partial t}=\operatorname{div}(K_0 a\nabla s)-(b_s'\mathbf{v}+\mathbf{F}_s')\nabla s+\operatorname{div}_x\mathbf{F}. \tag{3.7}$$

The energy equation corresponding to (3.7)

$$\frac{1}{2}\left(\Pi(t,\rho)-\Pi(0,\rho)+E(t,\rho)=\int_0^t((b_s'\mathbf{v}+\mathbf{F}_s')\nabla s,s)+\operatorname{div}_x\mathbf{F},s)_{B_\rho}+(K_0 a\nabla s\mathbf{n},s)_{\partial B_\rho}\right)d\tau$$

similar to the case of Theorems 1.2 and 2.1, leads to an inequality of the form (1.29). Analysis of the latter completes the proof of the theorem.

Note 3.1. A similar theorem is valid for the function s(t, x) = 1 − s(t, x).

Note 3.2. In the region $B_{\rho_0}\times(0,t_0)$, the corrected pressure satisfies the elliptic equation $\operatorname{div}(K(x,0)\nabla p+\mathbf{f}(x,0))=0$.

Note 3.3. Theorem 3.1 can give the following physical interpretation. Let the region B_{ρ_0} be occupied by only one fluid [s(0, x) = 0 or s(0, x) = 1] at the initial moment t = 0. Then for any action outside of B_{ρ_0}, the displacement of the given fluid from B_{ρ_0} does not begin before the moment of time $t_0 > 0$.

LITERATURE CITED

1. Ya. B. Zel'dovich and A. S. Kompaneets, "Theory of heat propagation with temperature-dependent conduction," in: Collected Articles in Honor of the Seventieth Birthday of Academician A. F. Ioffe [in Russian], Izd. Akad. Nauk SSSR, Moscow (1950).
2. G. I. Barenblatt, "Certain nonsteady motions of liquid and gas in a porous medium," Prikl. Mat. Mekh., 15, No. 1 (1952).
3. G. I. Barenblatt and M. I. Vishik, "Finite rate of propagation in problems of nonsteady filtration of liquid and gas," Prikl. Mat. Mekh., 20, No. 3 (1956).
4. L. V. Ovsyannikov, "Gas flow with a straight transition line," Prikl. Mat. Mekh., 13, No. 5 (1949).
5. A. S. Kalashnikov, "Certain problems of the qualitative theory of degenerate nonlinear second-order parabolic equations," Ukr. Mat. Zh., 42, No. 2(254) (1987).
6. L. K. Martinson, "Study of a mathematical model of nonlinear heat conduction in media with volume absorption," in: Mathematical Modeling. Processes in Nonlinear Media [Russian translation], Nauka, Moscow (1986).
7. A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, Regimes with Enhancement in Problems for Quasilinear Parabolic Equations [in Russian], Nauka, Moscow (1987).
8. S. N. Antontsev, "Character of perturbations described by the solutions of multidimensional degenerate parabolic equations," Din. Sploshnoi Sredy, 40, (1979).
9. S. N. Antontsev, "Localization of the solutions of degenerate nonlinear elliptic and parabolic equations," Dokl. Akad. Nauk SSSR, 260, No. 6 (1981).
10. S. N. Antontsev, "Finite rate of propagation of perturbations in multidimensional problems of two-phase diffusion," Proceedings of the Scientific Seminar LOMI, Akad. Nauk SSSR, Leningrad (1980), Vol. 96.
11. S. N. Antontsev, Localization of the Solutions of Degenerate Equations of Continuum Mechanics [in Russian], Inst. Geofiz. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1986).
12. J. I. Diaz and L. Veron, "Compacité du support des solutions d'équations quasilinéaires elliptiques," C. R. Acad. Sci. Paris, 297, 149 (1983).
13. J. I. Diaz and L. Veron, "Local vanishing properties of solutions of elliptic and parabolic quasilinear equations," Trans. Am. Math. Soc., 290, No. 2 (1985).
14. F. Bernis, "Compactness of the support for some nonlinear elliptic problems of arbitrary order in dimension N," Comm. Partial Diff. Equat., 9, No. 3 (1984).
15. F. Bernis, "Finite speed of propagation and asymptotic rates for some nonlinear higher-order parabolic equations with absorption," Proc. R. Soc. Edinburgh, 104A (1986).
16. I. B. Palymskii, "Some qualitative properties of solutions of equations of nonlinear heat conduction with absorption," ChMMSS, 16, No. 1 (1985).
17. Yu. G. Rykov, "Finite rate of propagation of perturbations for generalized solutions of degenerate quasilinear parabolic equations," in: Differential Equations and Their Applications [in Russian], Izd. Mosk. Gos. Univ., Moscow (1984).

18. Yu. G. Rykov, "Behavior over time of generalized degenerate parabolic equations with nonisotropic nonexponential nonlinearities," in: Differential Equations and Their Application [in Russian], Izd. Mosk. Gos. Univ., Moscow (1984).
19. S. N. Antontsev, "Localization of the solutions of certain degenerate equations of continuum mechanics," in: Problems of Mathematics and Mechanics [in Russian], Nauka, Novosibirsk (1983).
20. S. N. Antontsev and A. A. Panin, "Localization of the solutions of the equations of a viscous gas with viscosity dependent on density," Din. Sploshnoi Sredy, 86 (1988).
21. S. N. Antontsev, "Localization of the solutions of degenerate nonlinear elliptic equations," in: Equations in Partial Derivatives and Problems with a Free Boundary [in Russian], Naukova Dumka, Kiev (1983).
22. S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov, Boundary-Value Problems of the Mechanics of Inhomogeneous Fluids [in Russian], Nauka, Novosibirsk (1983).
23. S. N. Antontsev, G. P. Epikhov, and A. A. Kashevarov, Systematic Mathematical Modeling of Water Transport Processes [in Russian], Nauka, Novosibirsk (1986).
24. L. V. Ovsyannikov, Introduction to Continuum Mechanics [in Russian], Vol. 2, Novosibirsk Gos. Univ. (1977).
25. A. Kh. Mirzadzhanzade and P. M. Ogibalov, Transient Motions of Viscoplastic Media, Izd. Mosk. Gos. Univ., Moscow (1970).
26. P. P. Mosolov and V. P. Myasnikov, Mechanics of Rigid-Plastic Media [in Russian], Nauka, Moscow (1981).
27. M. Bohm, "On a nonhomogeneous non-Newtonian fluid," Rep./Lefschetz Center for Dynam. Systems. Division Appl. Mat., Brown Univ., Providence (1983), pp. 83-8.
28. J. L. Lions, Some Methods of Solving Nonlinear Boundary-Value Problems [Russian translation], Mir, Moscow (1972).
29. O. A. Ladyzhenskaya, Mathematical Aspects of the Dynamics of a Viscous Incompressible Fluid [in Russian], Nauka, Moscow (1970).
30. S. N. Antontsev, "Metastable localization of the solutions of degenerate parabolic equations of general form," Din. Sploshnoi Sredy, 83 (1987).
31. J. I. Diaz, Nonlinear Partial Differential Equations and Free Boundaries, Vol. 1, Elliptic Equations, Putnam (1985) (Putnam Res. Not. Math., 106).

STABILITY OF REGULAR SHOCK WAVE REFLECTION

V. M. Teshukov

UDC 533.6.011

As is known, the problem of steady supersonic inviscid gas flow around an infinite wedge has a nonunique solution [1]. One of the solutions determines the flow with a weak attached compression shock, and the other with a strong shock. An analogous nonuniqueness occurs in the problem of regular reflection of an oblique compression shock from a rigid wall (strong and weak reflected shocks). Stability of the flow with weak and strong reflected shocks relative to small nonstationary perturbations is investigated in this paper. Correctness of the problem of the perturbations of the flow with a weak reflected shock and incorrectness of the problem of perturbations of the flow with the strong shock are established. This result determines the stability boundary of regular shock reflection. Questions of the stability of flows with strong and weak shocks have long attracted the attention of researchers [2]. Analytic results were obtained earlier just for model simplified formulations of the gas dynamic perturbation problem [3-5]. Assertions about the stability of flows with weak shocks and the instability of flows with strong shocks were expressed in [5, 6] in connection with an analysis of the results of calculation experiments.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 26-33, March-April, 1989. Original article submitted August 1, 1988.

